Chapter 2

COMPLEX NUMBERS

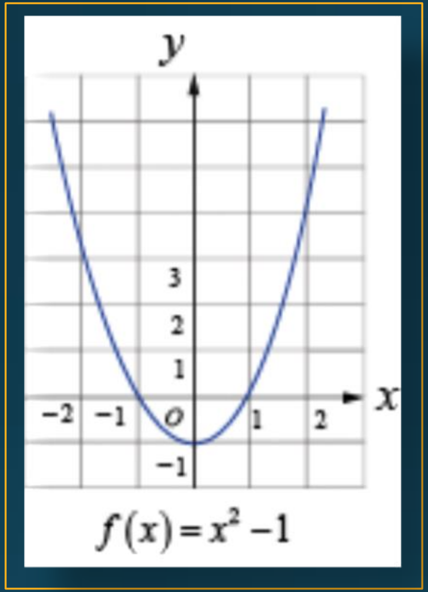
CONTENTS

2.1 Introduction to complex number

Upon Completion of this chapter, students will be able to:

- do algebraic operations on complex numbers
- identify the complex numbers in Argand plane
- find the conjugate and modulus of a complex number and to apply it
- find the polar form of a complex number and apply it
- apply de Moivre theorem to find the nth roots of complex numbers.

Before introducing complex numbers, let us try to answer the question "Whether there exists a real number whose square is negative?" Let's look at simple examples to get the answer for it. Consider the equations 1 and 2

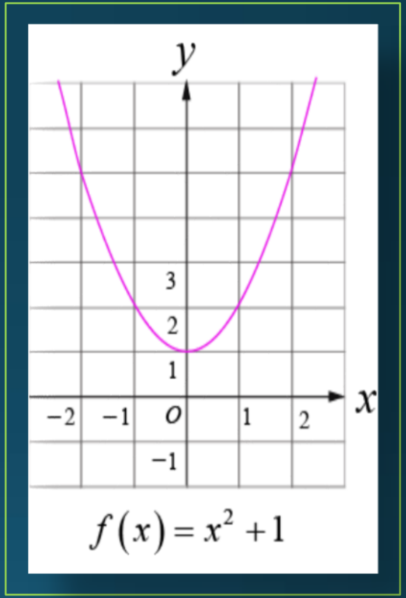


Equation 1

$$x^2-1=0$$
$$x=\pm\sqrt{1}$$
$$x=\pm1$$

Equation 2

$$x^2+1=0$$
$$x=\pm\sqrt{-1}$$
$$x=\pm ?$$



This is because, when we square a real number it is impossible to get a negative real number. If Equation 2 have solutions, then we must create an imaginary number as a square root of -1. This imaginary unit $\sqrt{-1}$ is denoted by i

2.2 Complex numbers

In this section, we define

(i) Complex numbers in rectangular form

(ii) Argand plane

(iii) Algebraic operations on complex numbers

2.2.1 Rectangular form of a complex number

Definition (Rectangular form of a complex number):

A complex number is of the form $x+iy$ (or) $x+yi$ *,where* x *and* y *are real numbers.* x *is called the real part and* y *is called the imaginary part of the complex number.*

If $x=0$, the complex number is said to be purely imaginary. If $y=0$, the complex number is called real. Zero is the only number which is at once real and purely imaginary. It is customary to denote the standard rectangular form of a complex number $x+iy$ as z and we write $x=\operatorname{Re}(z)$ and $y=\operatorname{Im}(z)$

Definition

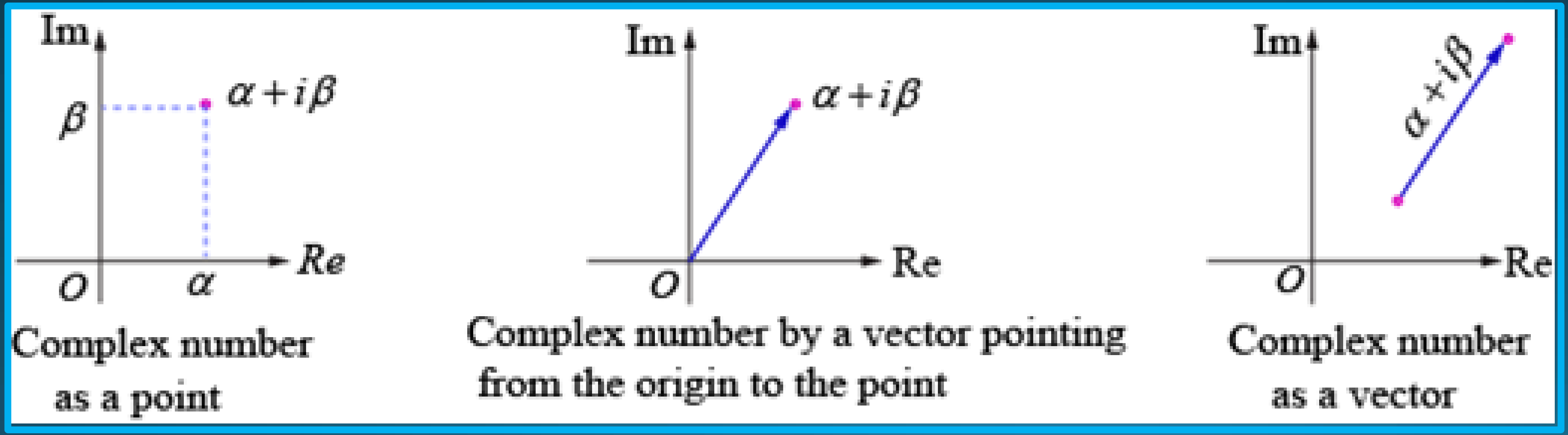
Two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are said to be equal if and only if

$$\operatorname{Re}(z_1) = \operatorname{Re}(z_2) \quad \text{and} \quad \operatorname{Im}(z_1) = \operatorname{Im}(z_2)$$

That is $x_1 = x_2$ and $y_1 = y_2$

2.2.2 Argand plane

A complex number is represented not only by a point, but also by a vector pointing from the origin to the point. The number, the point, and the vector will all be denoted by the same letter . As usual we identify all vectors which can be obtained from each other by parallel displacements. In this chapter denotes the set of all complex numbers. Geometrically, a complex number can be viewed as either a point or a vector in the Argand plane.



Complex number as a point

Complex number by a vector pointing from the origin to the point

Complex number as a vector

2.2.3 Algebraic operations on complex numbers

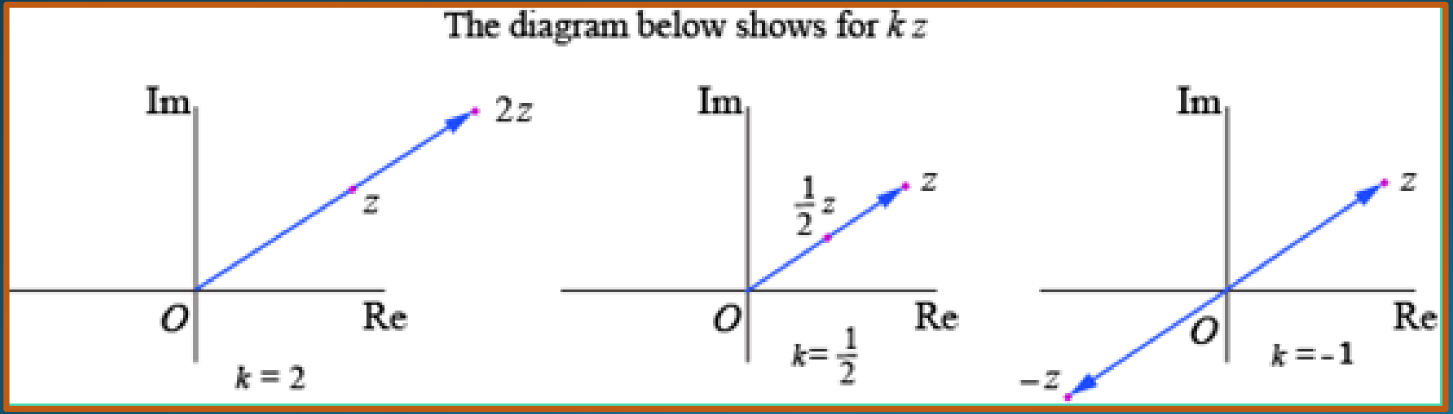
In this section, we survey the algebraic and geometric structure of the complex number system. We assume various corresponding properties of real numbers to be known.

(i) Scalar multiplication of complex numbers:

If $z = x + iy$ and $k \in \mathbb{R}$, then

$$kz = (kx) + (ky)i$$

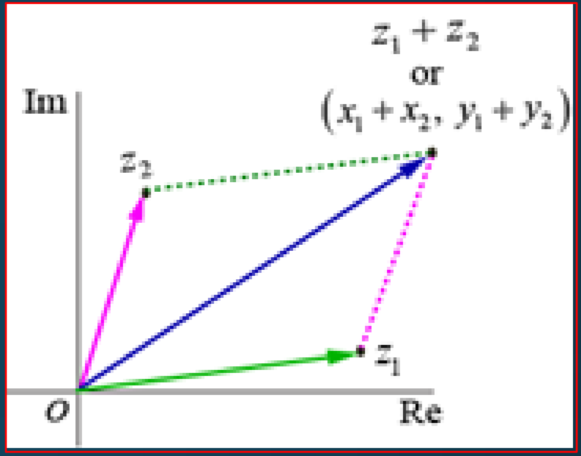
In particular $0z = 0$ and $(-1)z = -z$



(ii) Addition of complex numbers:

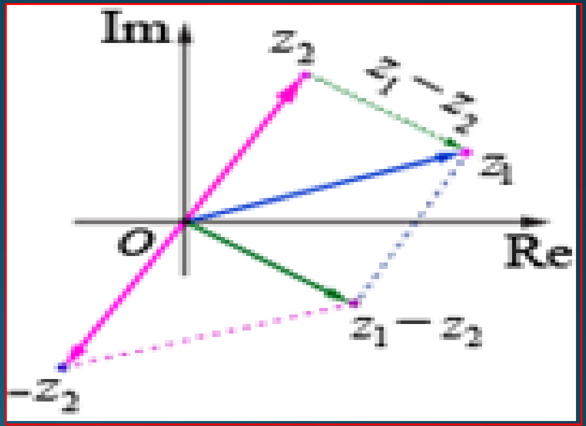
If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, and $x_1, x_2, y_1, y_2 \in R$

$$z_1 + z_2 = \left(x_1 + x_2\right) + i\left(y_1 + y_2\right)$$



(iii) Subtraction of complex numbers

$$z_1 - z_2 = \left(x_1 - x_2\right) + i\left(y_1 - y_2\right)$$

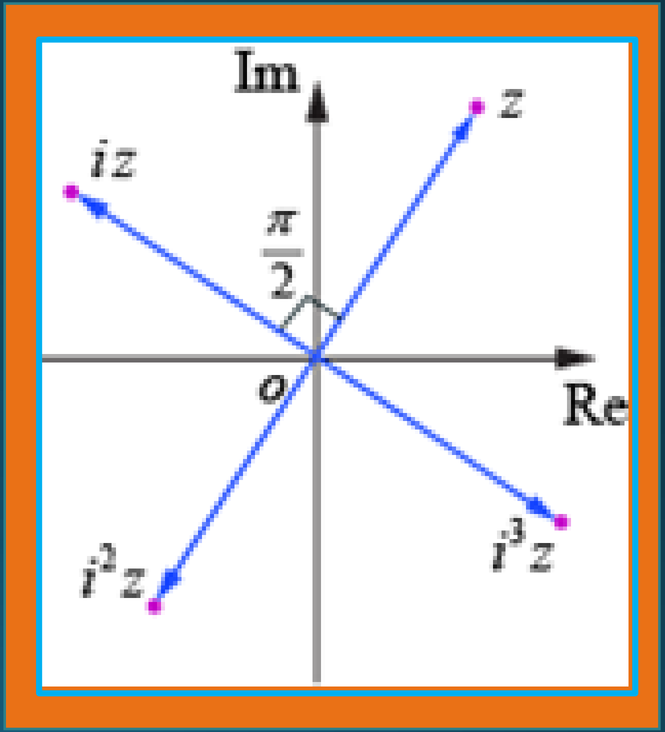


(iv) Multiplication of complex number

$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$$

Multiplication of complex number z by i

The complex number iz is a rotation of z by $90°$ or $\frac{\pi}{2}$ radians in the counter clockwise direction. In general, multiplication of a complex number z by i successively gives a $90°$ counter clockwise rotation successively about the origin



2.3. Basic algebraic properties of complex numbers

The complex numbers satisfy the following properties under addition.	The complex numbers satisfy the following properties under multiplication.
(*i*) Closure property	(*i*) Closure property
(*ii*) The commutative property	(*ii*) The commutative property
(*iii*) The associative property	(*iii*) The associative property
(*iv*) The additive identity The complex number *0* is known as additive identity	(*iv*) The Multiplicative identity The complex number 1 is known as multiplicative identity.
(*v*) The additive inverse -z is called the additive inverse of z .	(*v*) The Multiplicative inverse 1/z is the multiplicative inverse of z.

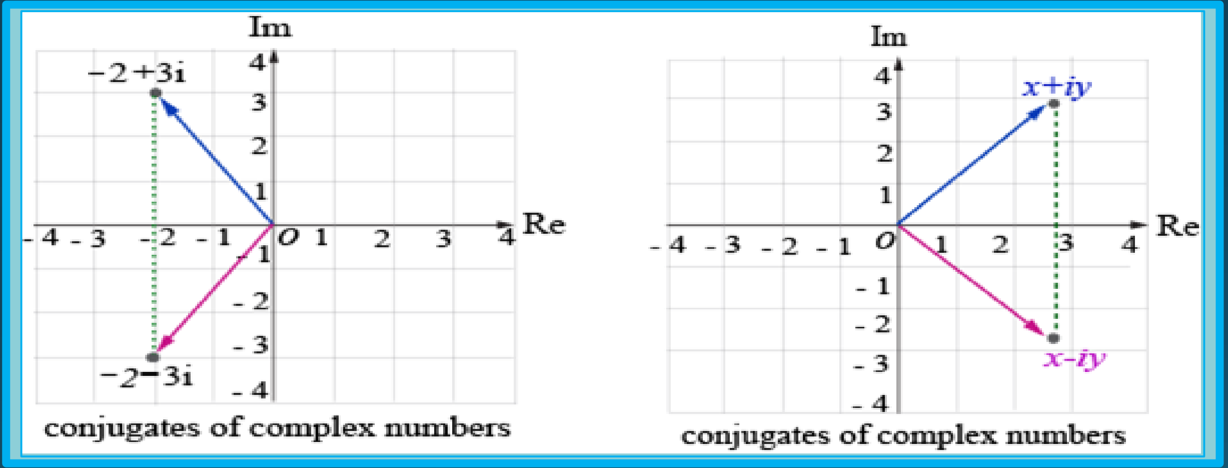
2.3. Basic algebraic properties of complex numbers

$x+iy$.

2.4 Conjugate of a complex number.

Definition: *Conjugate of the complex number* $x+iy$ *is defined as* $x-iy$.

2.4.1 Geometrical representation of conjugate of a complex number



Two complex numbers of the form and are complex conjugates to each other. The conjugate is useful in division of complex numbers. The complex number can be replaced with a real number in the denominator by multiplying the numerator and denominator by the conjugate of the denominator. This process is similar to rationalizing the denominator to remove surds.

2.4. 2 The properties of the complex conjugate

(1) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

(2) $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$

(3) $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$

(4) $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\overline{z_1}}{\overline{z_2}}, \quad z_2 \neq 0$

(5) $\operatorname{Re}(z) = \dfrac{z + \overline{z}}{2}$

(6) $\overline{\left(z^n\right)} = \left(\overline{z}\right)^n$, where n is an integer

(7) z is real if and only if $z = \overline{z}$

(8) z is purely imaginary if and only if $z = -\overline{z}$

(9) $\overline{\overline{z}} = z$

(10) $\operatorname{Im}(z) = \dfrac{z - \overline{z}}{2i}$

Show that

(i) $\left(2+i\sqrt{3}\right)^{10}-\left(2-i\sqrt{3}\right)^{10}$ is purely imaginary

(ii) $\left(\dfrac{19-7i}{9+i}\right)^{12}+\left(\dfrac{20-5i}{7-6i}\right)^{12}$ is real

Use the following properties to solve the above problems

z is real if and only if $z=\bar{z}$

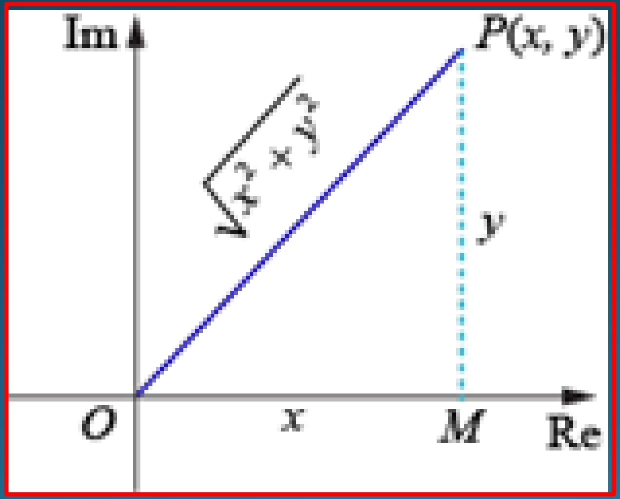
z is purely imaginary if and only if $z=-\bar{z}$

$z = x + iy$, $\sqrt{x^2+y^2}$

2.5 Modulus of a complex number

Definition

If $z = x + iy$, then $\sqrt{x^2 + y^2}$ is called modulus of z. It is denoted by $|z|$.



2.5.1 Properties of Modulus of complex number

(1) $|z| = |z|$

(2) $|z_1 + z_2| \le |z_1| + |z_2|$ (Triangle inequality)

(3) $|z_1 z_2| = |z_1||z_2|$

(4) $|z_1 - z_2| \ge \big||z_1| - |z_2|\big|$

(5) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}, \quad z_2 \ne 0$

(6) $|z^n| = |z|^n$ where n is an integer

(7) $\mathrm{Re}(z) \le |z|$

(8) $\mathrm{Im}(z) \le |z|$

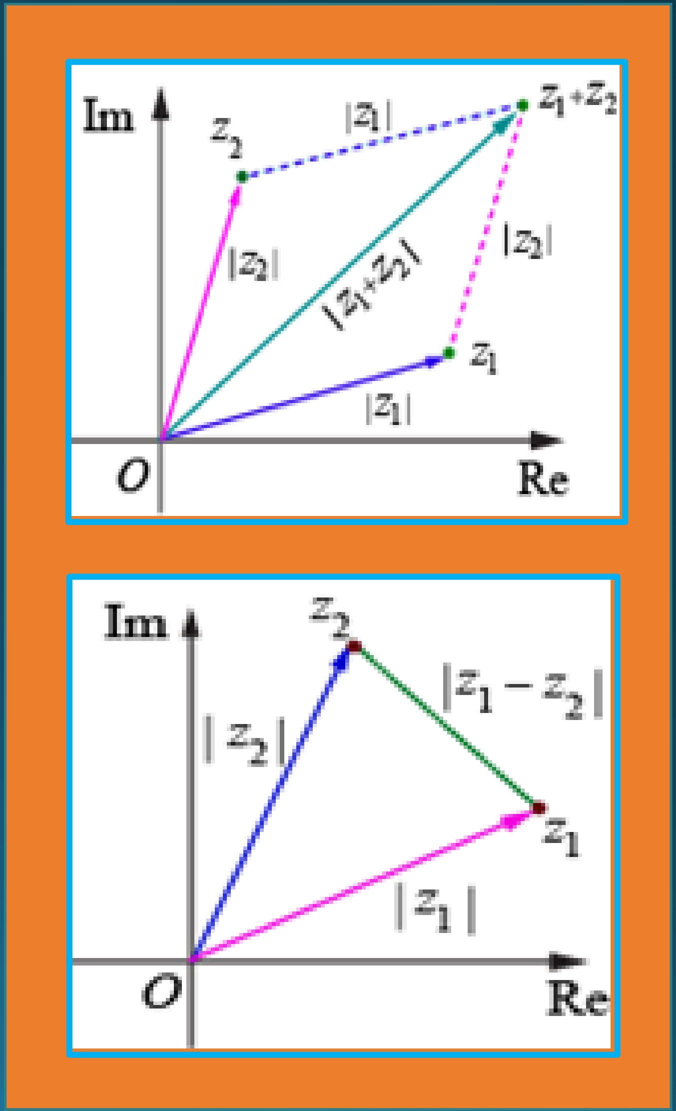
Triangle inequality

For any two complex numbers z_1 and z_2

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

To find the lower bound and upper bound use

$$\big||z_1| - |z_2|\big| \leq |z_1 + z_2| \leq |z_1| + |z_2|$$



2.5.2 Square roots of a complex number : $\sqrt{a+ib}$

Suppose the square root of $\sqrt{a+ib}$ is $x+iy=z$

$$\boxed{\sqrt{a+ib} = \pm\left(\sqrt{\frac{|z|+a}{2}} + i\frac{b}{|b|}\sqrt{\frac{|z|-a}{2}}\right) \quad , b \neq 0}$$

2.6 Geometry and locus of complex numbers

Equation of Complex Form of a Circle

The locus of z that satisfies the equation $|z - z_0| = r$
where z_0 is fixed complex number and r is a fixed positive real number
consists of all points whose distance from z_0 is r.
Therefore $|z - z_0| = r$ is complex form of the equation of a circle (see fig)

(i) If $|z - z_0| < r$, represents the points interior of the circle

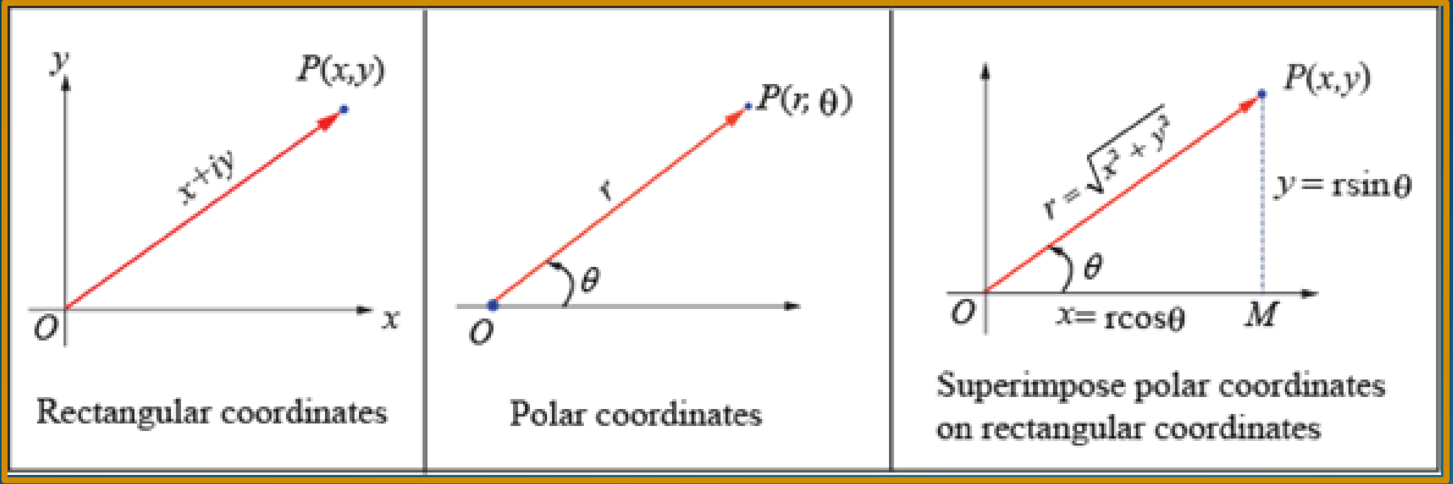
(ii) If $|z - z_0| > r$, represents the points exterior of the circle.

2.7 Polar form and Exponential form of a complex number

When performing addition and subtraction of complex numbers, we use rectangular form. This is because we just add real parts and add imaginary parts; or subtract real parts, and subtract imaginary parts. When performing multiplication or finding powers or roots of complex numbers, use an alternate form namely, polar form, because it is easier to compute in polar form than in rectangular form.

2.7.1 Polar form of a complex number

Polar coordinates form another set of parameters that characterize the vector from the origin to the point $z = x + iy$, with magnitude and direction. The polar coordinate system consists of a fixed point *O* called the pole and the horizontal half line emerging from the pole called the initial line (polar axis). If *r* is the distance from the pole to a point *P* and θ is an angle of inclination measured from the initial line in the counter clockwise direction to the line *OP*, then *r* and θ of the ordered pair (r, θ) are called the polar coordinates of *P*. superimposing this polar coordinate system on the rectangular coordinate system, as shown in diagram



we can write Polar form as $z = x + iy = r(\cos\theta + i\sin\theta) = r\,cis\,\theta.$

Definition :

Let r and θ be polar coordinates of the point $P(x, y)$
that corresponds to a non-zero complex number $z = x + iy$
The polar form or trigonometric form of a complex number is

$$z = r(\cos\theta + i\sin\theta)$$

The value r represents the absolute value, or modulus, of the complex number z. The angle θ is called the argument or amplitude of the complex number z denoted by $\theta = \arg(z)$.

(i) If $z = 0$, the argument θ is undefined; and so it is understood that $z \neq 0$ whenever polar coordinates are used.

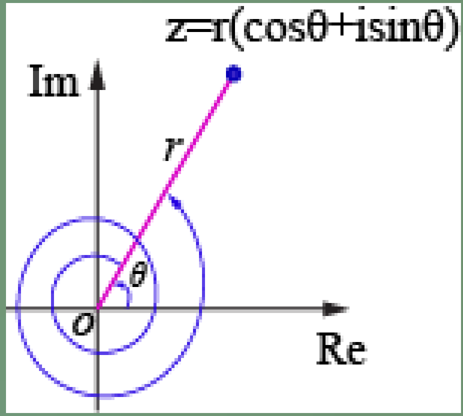
(ii) If the complex number $z = x + iy$ has polar coordinates (r, θ), its conjugate $\overline{z} = x - iy$ has polar coordinates $(r, -\theta)$.

Squaring and adding (1) and (2), and taking square root, the value of r is given by

$$r = |z| = \sqrt{x^2 + y^2}.$$

Dividing (2) by (1), $\dfrac{r\sin\theta}{r\cos\theta} = \dfrac{y}{x} \Rightarrow \tan\theta = \dfrac{y}{x}$

The real number θ represents the angle, measured in radians, that z makes with the positive real axis when z is interpreted as a radius vector. The angle θ has an infinitely many possible values, including negative ones that differ by integral multiples of 2π. Those values can be determined from the equation $\tan \theta = y/x$, where the quadrant containing the point corresponding to z must be specified. Each value of θ is called an *argument* of z, and the set of all such values is obtained by adding multiple of 2π to θ , and it is denoted by arg z.

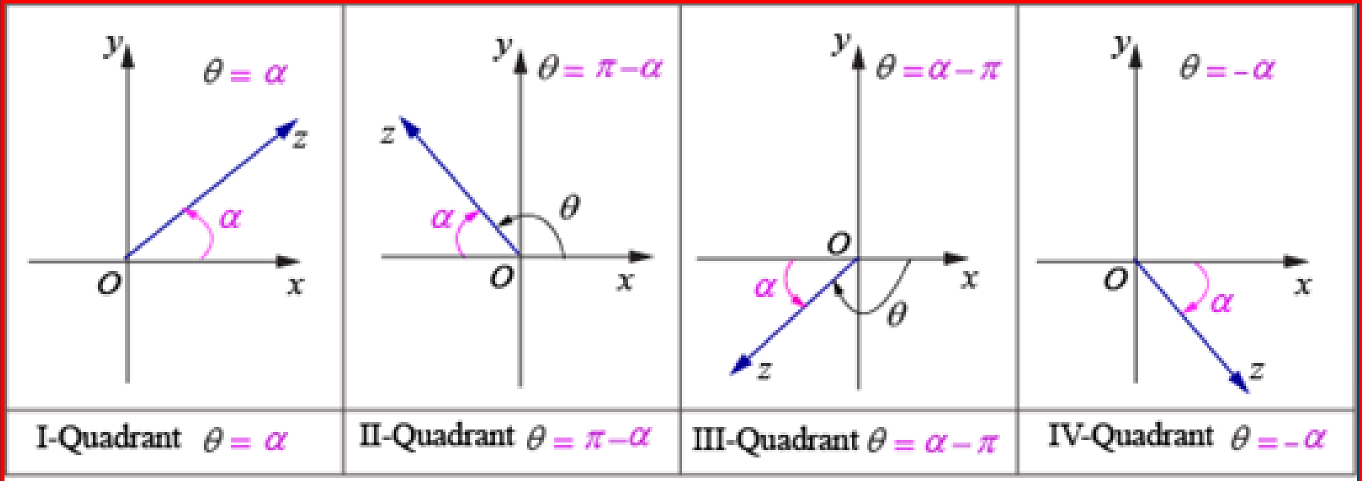


The *principal value* θ is denoted by Arg z,

$$-\pi < \mathrm{Arg}(z) \leq \pi \quad \text{or} \quad -\pi < \theta \leq \pi$$

The principal value ϑ is denoted by *Arg z*,

$$-\pi < \mathrm{Arg}(z) \leq \pi \quad \text{or} \quad -\pi < \theta \leq \pi$$



Principal Argument of a complex number

The capital A is important here to distinguish the principal value from the general value. Evidently, in practice to find the principal angle θ, we usually compute $\alpha = \tan^{-1}\left|\frac{y}{x}\right|$ and adjust for the quadrant problem by adding or subtracting α with π appropriately.

$$\arg z = Arg\ z + 2n\pi \quad (n = 0, \pm 1, \pm 2, \ldots).$$

The properties of arguments

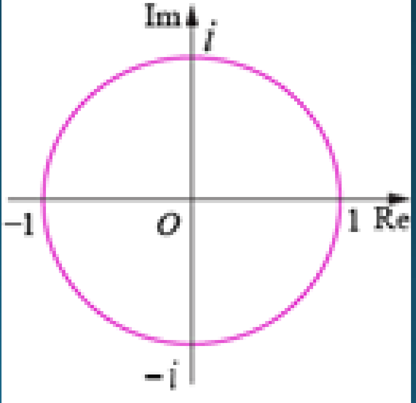
$$(1)\quad \arg\left(z_1\ z_2\right) = \arg z_1 + \arg z_2$$

$$(2)\quad \arg\left(\frac{z_1}{z_2}\right) = \arg z_1 + \arg z_2$$

$$(3)\quad \arg\left(z^n\right) = n\ \arg\ z$$

For instance the principal argument and argument of , and is shown below

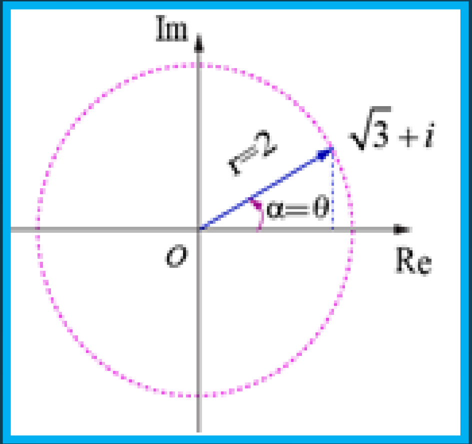
z	1	i	-1	$-i$
$Arg(z)$	0	$\frac{\pi}{2}$	π	$-\frac{\pi}{2}$
$\arg z$	$2n\pi$	$2n\pi + \frac{\pi}{2}$	$2n\pi + \pi$	$2n\pi - \frac{\pi}{2}$

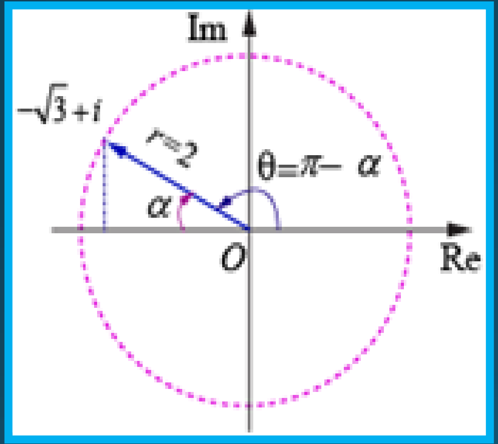


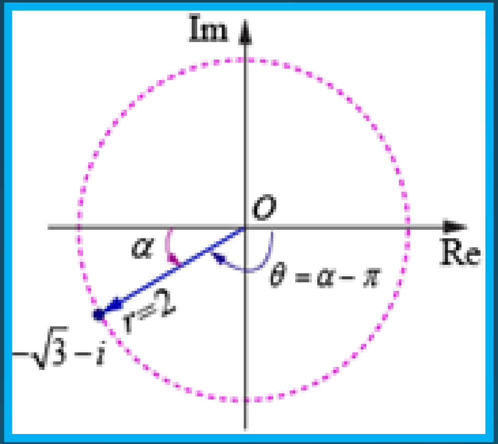
The modulus and principal argument of the following complex numbers.

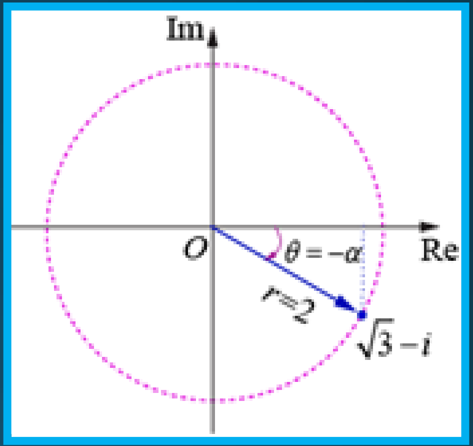
(i) $\sqrt{3}+i$ (ii) $-\sqrt{3}+i$ (iii) $-\sqrt{3}-i$ (iv) $\sqrt{3}-i$

The principal argument and modulus are shown below in figures









$\sqrt{3}+i$ $-\sqrt{3}+i$ $-\sqrt{3}-i$ $\sqrt{3}-i$

2.7.2 Euler's Form of the complex number:

The following identity is known as Euler's formula (phasor form)

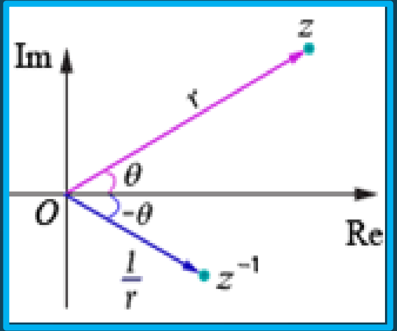
$$e^{i\theta} = \cos\theta + i\sin\theta$$

Euler formula gives the polar form $z = r\,e^{i\theta}$

2.7.3 Properties of polar form

Property 1

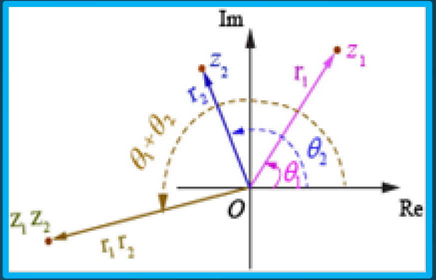
If $z = r(\cos\theta + i\sin\theta)$, then $z^{-1} = \frac{1}{r}(\cos\theta - i\sin\theta)$



Property 2

If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

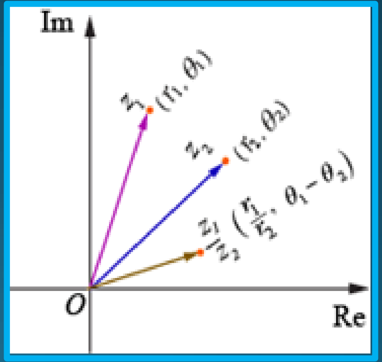
then $z_1 z_2 = r_1(\cos\theta_1 + i\sin\theta_1) r_2(\cos\theta_2 + i\sin\theta_2)$



Property 3

If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$,

then $\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$



2.8 de Moivre's Theorem and its applications

Abraham de Moivre (1667–1754) was one of the mathematicians to use complex numbers in trigonometry. The formula $(\cos\theta + i\sin\theta)^n = (\cos n\theta + i\sin n\theta)$ known by his name, was instrumental in bringing trigonometry out of the realm of geometry and into that of analysis.

de Moivre 1667–1754

de Moivre's Theorem:

Given any complex number $\cos\theta + i\sin\theta$ and any integer n,

$$(\cos\theta + i\sin\theta)^n = (\cos n\theta + i\sin n\theta)$$

Corollary of de Moivre's Theorem

$$(1)\quad (\cos\theta - i\sin\theta)^n = (\cos n\theta - i\sin n\theta)$$

$$(2)\quad (\cos\theta + i\sin\theta)^{-n} = (\cos n\theta - i\sin n\theta)$$

$$(3)\quad (\cos\theta - i\sin\theta)^{-n} = (\cos n\theta + i\sin n\theta)$$

$$(4)\quad \sin\theta + i\cos\theta = i\left(\cos\theta - i\sin\theta\right)$$

$z^n = 1$

2.8.3 The n roots of unity.

The solution of the equation $z^n = 1$, for positive values of integer *n* are the *n* roots of the unity.

Using deMoivre's theorem we can find the *n* roots of unity from the equation given below

$$z^{1/n} = \left(\cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right) \right) = e^{\frac{i2k\pi}{n}}, k = 0, 1, 2, 3, \ldots, n-1$$

The sum of all the n roots of unity

$$1 + \omega + \omega^2 + \cdots + \omega^{n-1} = 0$$

The product of all the n roots of unity

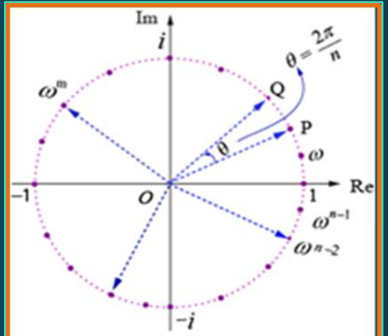
$$1\ \omega\ \omega^2 \ldots \omega^{n-1} = (-1)^{n-1}$$

(1) All the roots of n- th roots unity are in Geometrical Progression

(2) Sum of the n roots of n- *th* roots unity is always equal to zero.

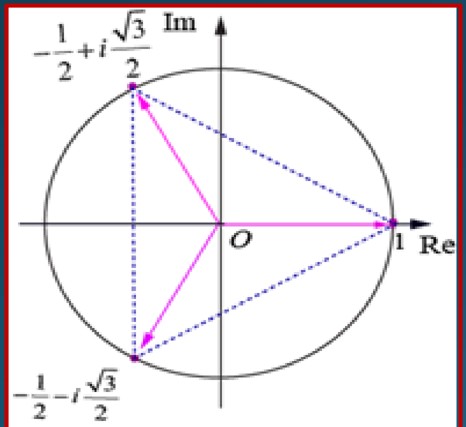
(3) Product of the roots of n-th roots unity is equal to .

(4) All the n roots of n-*th* roots unity lie on the circumference of a circle whose centre is at the origin and radius equal to 1 and these n roots divide the circle into n equal parts and form a polygon of n sides $(-1)^{n-1}$



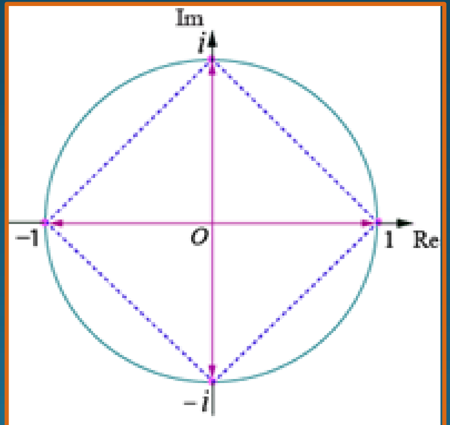
n roots of unity

The cube roots of unity.



Cubic roots of unity

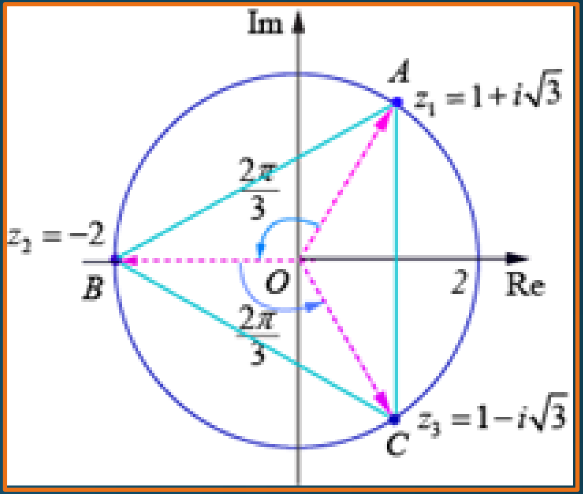
The Fourth roots of unity



Fourth roots of unity

The complex number $ze^{i\theta}$ is a rotation of z by θ radians in the counter clockwise direction about the origin

Problem: Suppose z_1, z_2, and z_3 are the vertices of an equilateral triangle inscribed in the circle $|z| = 2$. If $z_1 = 1 + i\sqrt{3}$, then find z_2 and z_3.



Solution: we can obtain z_2, and z_3 by the rotation of z_1 by $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$ respectively. That is

$$z_2 = z_1 e^{i\frac{2\pi}{3}} = \left(1 + i\sqrt{3}\right) e^{i\frac{2\pi}{3}} = -2$$

$$z_3 = z_1 e^{i\frac{4\pi}{3}} = \left(1 + i\sqrt{3}\right) e^{i\frac{4\pi}{3}} = 1 - i\sqrt{3}$$

If $z = 2 - 2i$, find the rotation of z by θ radians in the counter clockwise direction about the origin when

(i) $\theta = \frac{\pi}{3}$ (ii) $\theta = \frac{2\pi}{3}$ (iii) $\theta = \frac{3\pi}{2}$

In this chapter the letter is ω used for n th roots of unity. Therefore the value of ω is depending on n as shown in table.

value of n	2	3	4	5	$\ldots\cdots$	k
value of ω	$e^{i\frac{2\pi}{2}}$	$e^{i\frac{2\pi}{3}}$	$e^{i\frac{2\pi}{4}}$	$e^{i\frac{2\pi}{5}}$	$\ldots$	$e^{i\frac{2\pi}{k}}$